Multiple Integrals

Calculus I: what is the area under a curve $y = f(x)$ above an interval $[a, b]$ on x-axis?

Calculus III: what is the volume under a surface $z = f(x,y)$ above a rectangle R in xy-plane – say $[a,b] \times [c,d]$?

Solution: Slice the loaf parallel to the x-axis.
To find the volume of a slice, first butter it to find it area.

The area of one slice at a fixed value of y is $A = \int_{x=a}^{x=b} f(x,y)\,dx$, where we're treating y as a constant. (This is the reverse of taking $\frac{d}{dx}$)

you will get an expression depending on y, but there should be no x's – you plugged in the limits for x.

Volume of that slice = (area) width $= \left[\int_{x=a}^{x=b} f(x,y)\,dx\right] dy$

Volume of the whole loaf

$$\int_{y=c}^{y=d} \left[\int_{x=c}^{x=b} f(x,y)\,dx\right] dy$$

remember, you get rid of all the x's by doing the intergal!

Reminder: write your variables in Calculas III.
At times we'll use $x, y, z, r, \theta, \rho, \phi, u, v$, etc.

Ex] Find the volume under the bowl

$f(x,y) = x^2+y^2$ & above the rectangle

$R = \underbrace{[1,2]}_{x} \times \underbrace{[3,4]}_{y}$

$$\int_3^4 \left[\int_1^2 (x^2+y^2)\,dx \right] dy = \frac{44}{3}$$

- $$\int_{y=2}^{y=6} \int_{x=1}^{x=3} x e^{xy}\,dx\,dy$$

- $$\int_{x=1}^{x=3} \int_{y=2}^{y=6} x e^{xy}\,dy\,dx$$

$$\int_{x=1}^{x=3} e^{xy}\Big|_{y=2}^{y=6} dx = = \int_1^3 e^{6x} - e^{2x}\,dx$$

$$= \frac{e^{6x}}{6} - \frac{e^{2x}}{2}\Big|_1^3 = \frac{e^{18}}{6} - \frac{e^6}{3\cdot 2} - \frac{e^6}{6} + \frac{e^2}{2}$$

$$= \frac{e^{18}}{6} - \frac{4e^6}{6} - \frac{e^2}{2}$$

$$= \frac{e^{18}}{6} - \frac{2}{3}e^6 - \frac{e^2}{2}$$

$$\iint_R x\cos(xy)\,dA \quad \text{where} \quad R = [4,6] \times [2,3]$$

$$\int_4^6 \int_2^3 x \cos xy\,dy\,dx = \int_4^6 \sin xy\Big|_2^3\,dx$$

$$\int_4^6 \sin 3x - \sin 2x \, dx$$

$$= \frac{\cos 2x}{2} - \frac{\cos 3x}{3} \Big|_4^6$$

$$= \frac{3\cos 12}{3\cdot 2} - \frac{\cos 18}{3} - \frac{\cos 8}{2} + \frac{2\cos 12}{2\cdot 3}$$

$$= \frac{5}{6}\cos 12 - \frac{\cos 18}{3} - \frac{\cos 8}{2}$$

Ex] Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ & below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$.

$$\iint_R 10 + x^2 + 3y^2 \, dA = \int_0^1 \int_0^2 10 + x^2 + 3y^2 \, dy \, dx = \frac{86}{3}$$